Sequences and Series 4

1. By using the difference method find the sim of the first n terms:

$$\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{2\cdot3\cdot4\cdot5}+\frac{1}{3\cdot4\cdot5\cdot6}+\cdots=\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$$

Let $v_r=\frac{1}{3(r+1)(r+2)(r+3)}, u_r=\frac{1}{r(r+1)(r+2)(r+3)}$

$v_{r-1}-v_r=\frac{1}{3r(r+1)(r+2)}-\frac{1}{3(r+1)(r+2)(r+3)}=\frac{(r+3)-r}{3r(r+1)(r+2)(r+3)}=\frac{1}{r(r+1)(r+2)(r+3)}=u_r$

$\sum_{r=1}^{n}u_r=\sum_{r=1}^{n}(v_{r-1}-v_r)=v_0-v_n=\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$

2. Express $\frac{2}{r^2-1}$ in partial fractions. Hence, find a simple expression for $S=\sum_{r=2}^{n}\frac{2}{r^2-1}$ and determine whether S converges when n tends to infinity.

$\frac{2}{r^2-1}=\frac{1}{r-1}-\frac{1}{r+1}$

$S=\sum_{r=2}^{n}\frac{2}{r^2-1}=\sum_{r=2}^{n}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)=\frac{1}{2-1}-\frac{1}{n+1}=1-\frac{1}{n+1}$

$\sum_{r=2}^{\infty}\frac{2}{r^2-1}=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=\mathbf{1}$

3. If $\sum_{r=1}^{n}T_r=3n^2+4n$, find the value of $\sum_{r=1}^{n-1}T_r$.
Deduce the general term T_n and hence find $\sum_{r=n+1}^{2n}T_r$.

$\sum_{r=1}^{n-1}T_r=3(n-1)^2+4(n-1)=3n^2-2n-1$
$T_n=\sum_{r=1}^{n}T_r-\sum_{r=1}^{n-1}T_r=(3n^2+4n)-(3n^2-2n-1)=6n+1$
$\sum_{r=n+1}^{2n}T_r=\sum_{r=1}^{2n}T_r-\sum_{r=1}^{n}T_r=[3(2n)^2+4(2n)]-(3n^2+4n)=9n^2+4n$

4. The sum of the first 2n terms of a series $18n-12n^2$. Find the sum of the first n terms and the nth term of this series. Show that this series is an arithmetic.

$S(2n)=18n-12n^2=9(2n)-3(2n)^2$
Hence $S(n)=9n-3n^2$.
$T(n)=S(n)-S(n-1)=[9n-3n^2]-[9(n-1)-3(n-1)^2]=12-6n$
$T(n)-T(n-1)=[12-6n]-[12-6(n-1)]=-6$ which is a constant.
Therefore, this series is an arithmetic with a common different -6.

5. In a set of integers between the numbers 1 and 10,000,

(a) how many of these numbers are divisible by 3,4, 5 and 11?

(b) how many of these numbers are divisible by 3,4, 5 or 11?

(a) The number must be divisible by the least common multiple of 3,4, 5 and 11, which is 660.

Denote $\lfloor x \rfloor$ be the greatest integer smaller or equal to x .

Then the number between the numbers 1 and 10,000 which are divisible by 3,4, 5 and 11

$= \left\lfloor \frac{10000}{660} \right\rfloor = \mathbf{15}$

(b) Let A = { integers from 1 through 10000 that are multiples of 3 }

B = { integers from 1 through 10000 that are multiples of 4 }

C = { integers from 1 through 10000 that are multiples of 5 }

D= { integers from 1 through 10000 that are multiples of 11 }

Denote $\lfloor x \rfloor$ be the greatest integer smaller or equal to x . We have :

$|A| = \left\lfloor \frac{10000}{3} \right\rfloor = 3333$, $|B| = \left\lfloor \frac{10000}{4} \right\rfloor = 2500$, $|C| = \left\lfloor \frac{10000}{5} \right\rfloor = 2000$, $|D| = \left\lfloor \frac{10000}{11} \right\rfloor = 909$,

$|A \cap B| = \left\lfloor \frac{10000}{12} \right\rfloor = 833$, $|A \cap C| = \left\lfloor \frac{10000}{15} \right\rfloor = 666$, $|A \cap D| = \left\lfloor \frac{10000}{33} \right\rfloor = 303$,

$|B \cap C| = \left\lfloor \frac{10000}{20} \right\rfloor = 500$, $|B \cap D| = \left\lfloor \frac{10000}{44} \right\rfloor = 227$, $|C \cap D| = \left\lfloor \frac{10000}{55} \right\rfloor = 181$

$|A \cap B \cap C| = \left\lfloor \frac{10000}{60} \right\rfloor = 166$, $|A \cap B \cap D| = \left\lfloor \frac{10000}{132} \right\rfloor = 75$, $|A \cap C \cap D| = \left\lfloor \frac{10000}{165} \right\rfloor = 60$,

$|B \cap C \cap D| = \left\lfloor \frac{10000}{220} \right\rfloor = 45$

$|A \cap B \cap C \cap D| = \left\lfloor \frac{10000}{660} \right\rfloor = 15$

By the Principle of Exclusion and Inclusion,

Required number

$$= (3333 + 2500 + 2000 + 909) - (833 + 666 + 303 + 500 + 227 + 181)$$
$$-(166 + 75 + 60 + 45) + 15 = \mathbf{5701}$$

6. The sum of the first n terms of a geometric sequence is given by $S_n = 15(1 - 3^{-n})$. Find

(a) the nth term,

(b) the common ratio,

(c) the smallest value of n such that $S_\infty - S_n < 0.01$.

(a) $T_n = S_n - S_{n-1} = 15(1 - 3^{-n}) - 15\left(1 - 3^{-(n-1)}\right) = 15\left(3^{-(n-1)} - 3^{-n}\right)$

$= 15(3^{1-n} - 3^{-n}) = 15[3(3^{-n}) - 3^{-n}] = 15[2(3^{-n})] = \mathbf{30(3^{-n})}$

(b) $r = \frac{T_n}{T_{n-1}} = \frac{30(3^{-n})}{30\left(3^{-(n-1)}\right)} = \frac{1}{3}$

(c) $S_\infty = \frac{T_1}{1-r} = \frac{30(3^{-1})}{1-\frac{1}{3}} = 15$

$S_\infty - S_n < 0.01 \Rightarrow 15 - 15(1-3^{-n}) < 0.01 \Rightarrow 15(3^{-n}) < 0.01 \Rightarrow 3^n > 1500$

$\Rightarrow n\, log\ 3 > \log 1500 \Rightarrow n > \frac{\log 1500}{\log 3} = 6.65678$

$\therefore n = 7$

7. Verify the identity $\frac{2r-1}{r(r-1)} - \frac{2r+1}{r(r+1)} = \frac{2}{(r-1)(r+1)}$. Hence, using the method of difference, prove that

$\sum_{r=2}^{n} \frac{2}{(r-1)(r+1)} = \frac{3}{2} - \frac{2n+1}{n(n+1)}$.

Deduce the sum of the infinite series $\frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \cdots + \frac{1}{(n-1)(n+1)} + \cdots$

$$\frac{2r-1}{r(r-1)} - \frac{2r+1}{r(r+1)} = \frac{(2r-1)(r+1)-(r-1)(2r+1)}{r(r-1)(r+1)} = \frac{2r}{r(r-1)(r+1)} = \frac{2}{(r-1)(r+1)}$$

$$\sum_{r=2}^{n} \frac{2}{(r-1)(r+1)} = \sum_{r=2}^{n} \left[\frac{2r-1}{r(r-1)} - \frac{2r+1}{r(r+1)}\right] = \sum_{r=2}^{n} \left[\frac{2r-1}{r(r-1)}\right] - \sum_{r=2}^{n} \left[\frac{2r+1}{r(r+1)}\right]$$

$$= \left\{\frac{3}{2} + \sum_{r=3}^{n} \left[\frac{2r-1}{r(r-1)}\right]\right\} - \left\{\sum_{r=2}^{n-1} \left[\frac{2r+1}{r(r+1)}\right] + \frac{2n+1}{n(n+1)}\right\} = \frac{3}{2} - \frac{2n+1}{n(n+1)} + \left\{\sum_{r=3}^{n} \left[\frac{2r-1}{r(r-1)}\right] - \sum_{r=2}^{n-1} \left[\frac{2r+1}{r(r+1)}\right]\right\}$$

$$= \frac{3}{2} - \frac{2n+1}{n(n+1)} + \left\{\sum_{r=3}^{n} \left[\frac{2r-1}{r(r-1)}\right] - \sum_{k=3}^{n} \left[\frac{2k-1}{k(k-1)}\right]\right\} \text{ where } r = k-1$$

$$= \frac{3}{2} - \frac{2n+1}{n(n+1)}$$

$$\frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \cdots + \frac{1}{(n-1)(n+1)} = \sum_{r=2}^{n} \frac{1}{(r-1)(r+1)} = \frac{1}{2}\left[\frac{3}{2} - \frac{2n+1}{n(n+1)}\right]$$

$$\therefore \frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \cdots + \frac{1}{(n-1)(n+1)} + \cdots = \sum_{r=2}^{\infty} \frac{1}{(r-1)(r+1)} = \lim_{n\to\infty} \frac{1}{2}\left[\frac{3}{2} - \frac{2n+1}{n(n+1)}\right] = \frac{3}{4}$$

8. If $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are three consecutive terms in an arithmetic progression.

Show that $\frac{y+z}{x}, \frac{z+x}{y}, \frac{x+y}{z}$ also from three consecutive terms in an arithmetic progression.

Since $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

therefore $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}$

$(x+y+z)\left(\frac{1}{x} + \frac{1}{z}\right) = (x+y+z)\left(\frac{2}{y}\right)$

$\frac{x+y+z}{x} + \frac{x+y+z}{z} = \frac{2(x+y+z)}{y}$

$1+\frac{y+z}{x}+1+\frac{x+y}{z}=2+2\left(\frac{z+x}{y}\right)$

$\frac{y+z}{x}+\frac{x+y}{z}=2\left(\frac{z+x}{y}\right)$

Therefore $\frac{y+z}{x}, \frac{z+x}{y}, \frac{x+y}{z}$ also in A.P.

9. If S is the sum of the series $1+3x+5x^2+7x^3+\cdots+(2n+1)x^n$, for $x \neq 1$, by considering $(1-x)S$, or otherwise, show that $S=\frac{1+x-(2n+3)x^{n+1}+(2n+1)x^{n+2}}{(1-x)^2}, x \neq 1$.

Method 1

$$
\begin{aligned}
(1-x)S &= (1-x)[1+3x+5x^2+7x^3+\cdots+(2n+1)x^n] \\
&= 1+3x+5x^2+7x^3+\cdots+(2n+1)x^n \\
&\qquad -x-3x^2-5x^3-\cdots-(2n-1)x^n-(2n+1)x^{n+1} \\
&= 1+2x+2x^2+2x^3+\cdots+2x^n \qquad -(2n+1)x^{n+1} \\
&= 1+(2x)\left(\frac{1-x^n}{1-x}\right)-(2n+1)x^{n+1} \\
&= \frac{(1-x)+2x-2x^{n+1}-(1-x)(2n+1)x^{n+1}}{1-x} \\
&= \frac{1+x-(2n+3)x^{n+1}+(2n+1)x^{n+2}}{1-x}
\end{aligned}
$$

$\therefore S=\frac{1+x-(2n+3)x^{n+1}+(2n+1)x^{n+2}}{(1-x)^2}, x \neq 1$

Method 2

$S=1+3x+5x^2+7x^3+\cdots+(2n+1)x^n$

Replace $x=y^2$

$$
\begin{aligned}
S &= 1+3y^2+5y^4+7y^6+\cdots+(2n+1)y^{2n} \\
&= \frac{d}{dy}[y+y^3+y^5+\cdots+y^{2n+1}]=\frac{d}{dy}\left[\frac{y(1-y^{2n+2})}{1-y^2}\right]=\frac{1+y^2-(2n+3)y^{2n+2}+(2n+1)y^{2n+4}}{(1-y^2)^2} \\
&= \frac{1+x-(2n+3)x^{n+1}+(2n+1)x^{n+2}}{(1-x)^2}
\end{aligned}
$$

Method 3

$$
\begin{aligned}
S(1-x)^2 &= (1-x)^2[1+3x+5x^2+7x^3+\cdots+(2n+1)x^n] \\
&= (1-2x+x^2)[1+3x+5x^2+7x^3+\cdots+(2n+1)x^n] \\
&= 1+3x+5x^2+7x^3+\ \ldots\ +(2n+1)x^n \\
&\qquad -2x-6x^2-10x^3-\cdots-2(2n-1)x^n-2(2n+1)x^{n+1} \\
&\qquad\qquad +x^2+3x^3+\cdots.+(2n-3)x^n+(2n-1)x^{n+1}+(2n+1)x^{n+2} \\
&= 1+x-(2n+3)x^{n+1}+(2n+1)x^{n+2}
\end{aligned}
$$

$\therefore S=\frac{1+x-(2n+3)x^{n+1}+(2n+1)x^{n+2}}{(1-x)^2}, x \neq 1$

10. **(a)** The sequence of positive integers is grouped into four as follows:

$(1,2,3,4), (5,6,7,8), (9,10,11,12), \ldots$

Show that the sum of all integers in the kth bracket is $S_k = 2(8k-3)$.

(b) If the integers are similarly grouped with m integers in each bracket, find the sum S_n of all integers in the nth bracket in terms of m and n.

Hence, show that S_n, S_{2n}, S_{3n} are in arithmetic progression.

(a) Last term in the kth bracket is $4k$.

First term in the kth bracket is $4k-3$.

$S_k = (4k-3) + (4k-2) + (4k-1) + 4k = 16k - 6 = 2(8k-3)$

(b) The sequence is

$(1,2,\ldots,m), (m+1, m+2, \ldots, 2m), (2m+1, 2m+2, \ldots, 3m), \ldots$

Last term in the nth bracket is $a = mn$.

First term in the nth bracket is $l = mn - (m-1) = mn - m + 1$.

$$S_n = \frac{m}{2}(a+l) = \left\{\frac{m}{2}\left[[mn-(n+1)] + mn\right]\right\} = \frac{m}{2}(2mn - m + 1)$$

$$S_{2n} = \frac{m}{2}(4mn - m + 1)$$

$$S_{3n} = \frac{m}{2}(6mn - m + 1)$$

$$S_{2n} - S_n = \frac{m}{2}(4mn - m + 1) - \frac{m}{2}(a+l) = m^2 n$$

$$S_{3n} - S_{2n} = \frac{m}{2}(6mn - m + 1) - \frac{m}{2}(4mn - m + 1) = m^2 n$$

Since $S_{3n} - S_{2n} = S_{2n} - S_n$, hence S_n, S_{2n}, S_{3n} are in arithmetic progression.

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